

Support Theorems and an Injectivity Result for Integral Moments of a Symmetric m -Tensor Field

Anuj Abhishek and Rohit Kumar Mishra

1 Introduction

Let (Ω, g) be a compact, simple, real-analytic Riemannian manifold of dimension n with smooth boundary. We will parametrize the maximal geodesics in M with endpoints on $\partial\Omega$ by their starting points and directions.

Set

$$\Gamma_- := \{(x, \xi) \in TM \mid x \in \partial\Omega, |\xi| = 1, \langle \xi, \nu(x) \rangle < 0\},$$

where $\nu(x)$ is the outer unit normal to $\partial\Omega$ at x . Then we will define the q -th integral moment of a symmetric m -tensor field f , $I^q f$ as a function on Γ_- by

$$I^q f(x, \xi) = \int_0^{l(\gamma_{x,\xi})} t^q \langle f(\gamma_{x,\xi}(t)), \dot{\gamma}_{x,\xi}^m(t) \rangle dt = \int_0^{l(\gamma_{x,\xi})} t^q f_{i_1 \dots i_m}(\gamma_{x,\xi}(t)) \dot{\gamma}_{x,\xi}^{i_1}(t) \cdots \dot{\gamma}_{x,\xi}^{i_m}(t) dt.$$

where $\gamma_{x,\xi}(t)$ is the geodesic starting from x in the direction ξ and $l(\gamma_{x,\xi})$ is the value of the parameter t at which this geodesic intersects the boundary again. The above definition of integral moments for a symmetric m -tensor fields was first introduced by Sharafutdinov in the context of \mathbb{R}^n , see [16]. In the same paper he proved that if the first $(m+1)$ integral moments $I^q f$ for $q = 0, 1, \dots, m$ of a compactly supported symmetric m -tensor field f are known along all straight lines, then f can be uniquely recovered.

0-th integral moment coincides with the usual geodesic ray transform of a symmetric m -tensor field. In this work, we are interested in injectivity results and support theorem for integral moments defined above. Microlocal techniques play a very crucial role in proving such results. Guillemin first introduced the microlocal approach in the Radon transform setting, see [7]. Analytic microlocal techniques were used by Boman and Quinto in [3] to prove support theorems for Radon transforms with positive real-analytic weights. For more literature on such support theorems, we refer to the reader [13, 1, 4, 2, 5, 6, 14, 23] and references therein. For the Analytic microlocal techniques used in this paper, we will mainly refer to [20, 19, 21, 18, 11].

The geodesic ray transform of any symmetric tensor field of order 2, which in our notation will be denoted by $I^0(f)$ arises naturally in the context of lens and boundary rigidity problems and has been studied in e.g. [17],[15],[19],[20]. Support theorems for such transforms have

been of independent interest among mathematicians. In [20], the authors prove a s-injectivity result for symmetric 2 tensors fields. The same proof works for a symmetric tensor field of any order. That is if $I^0(f) = 0$ for all the geodesics of M then its solenoidal part vanishes. A question arises as to what data is sufficient for us to conclude such an injectivity result for the tensor field f itself. Using the result stated above, we show that if $I^q f = 0$ for $q = 0, 1, \dots, m$ for all the geodesics of M , then $f = 0$. Injectivity result for the local geodesic ray transform of a function has been proved in [22] using new techniques. We also treat the case in which the integral moments are known for the open set of geodesics that do not intersect a given geodesically convex set. We do so using the techniques laid out in [11], where the authors prove a Helgason type support theorem for symmetric tensor fields of order 2 over simple, real analytic Riemannian manifolds. We first extend the result in [11] for symmetric m tensor fields. Using this new result, we prove a stronger version of such support type theorems, i.e. if we know $I^q f = 0$ for $q = 0, 1, \dots, m$ over the open set of geodesics not intersecting a convex set, then it implies that the support of f lies in the convex set. We would also like to mention that Krishnan already proved such a support theorem for the case of functions in [9].

The paper is organized as follows. In section 2 we give the definitions and our main theorems. Section 3 has some preliminary Propositions and Lemmas that are needed for the proof of the main Theorems. In section 4 we will prove a Helgason type support theorem which we state in section 2 and prove the support theorem. In section 5 we prove the s-injectivity result mentioned above and use it to prove the injectivity of integral moments. We will provide proof of some Lemmas and inequalities in the Appendix.

Acknowledgements : We would like to thank Vladimir Sharafutdinov for suggesting this problem. Besides, we would also like to express our sincere gratitude to Eric Todd Quinto and Venky Krishnan for several hours of fruitful discussions.

2 Definitions and Statements of the Theorems

Definition 1 (Simple Manifold). *A compact Riemannian manifold (Ω, g) with boundary is said to be simple if*

- (i) *The boundary $\partial\Omega$ is strictly convex: $\langle \nabla_\xi \nu(x), \xi \rangle > 0$ for each $\xi \in T_x(\partial\Omega)$ where $\nu(x)$ is the unit outward normal to the boundary.*
- (ii) *The map $\exp_x : \exp_x^{-1}(M) \rightarrow M$ is a diffeomorphism for each $x \in M$.*

The second condition ensures that any two points x, y in M are connected by a unique geodesic in M that depends smoothly on x, y . Any simple manifold M is necessarily diffeomorphic to a ball in \mathbb{R}^n , see [17]. Therefore, in the analysis of simple manifolds, we can assume that M is a domain $\Omega \subset \mathbb{R}^n$. We are going to work on a fixed simple Riemannian manifold (Ω, g) with a fixed real analytic atlas. A tensor field is said to be analytic on a set U if it is real analytic in some neighborhood of U . Let $S^m(M)$ be the collection of symmetric m -tensor fields defined on M . We will work with symmetric m -tensor field $f = \{f_{i_1 \dots i_m}\}$.

We will assume the Einstein summation convention and raise and lower indexes using the metric tensor. The tensors $f_{i_1 \dots i_m}$ and $f^{i_1 \dots i_m} = f_{j_1 \dots j_m} g^{i_1 j_1} \dots g^{i_m j_m}$ will be thought of as same tensors with different representations.

It is well known from [17] that any symmetric m -tensor field can be decomposed uniquely in following way:

Theorem 1. [17, Theorem 3.3.2] *Let Ω be a compact Riemannian manifold with boundary; let $k \geq 1$ and $m \geq 0$ be integers. For every field $f \in H^k(S^m(\Omega))$, there exist uniquely determined $f^s \in H^k(S^m(\Omega))$ and $v \in H^{k+1}(S^{m-1}(\Omega))$ such that*

$$f = f^s + dv, \quad \delta f^s = 0, \quad v|_{\partial\Omega} = 0.$$

We call the fields f^s and dv the solenoidal and potential parts respectively of the field f .

Let $\tilde{\Omega}$ be an open, real analytic extension of Ω such that g can also be extended to a real analytic metric in $\tilde{\Omega}$. We will also extend all symmetric tensor fields f defined on Ω by 0 in $\tilde{\Omega} \setminus \Omega$. We will think of each maximal geodesic in Ω as a restriction of a geodesic with distinct endpoints in $\tilde{\Omega} \setminus \Omega$ to Ω . Let $\gamma_{[x,y]}$ be the geodesic connecting x and y .

Let \mathcal{A} be an open set of geodesics with endpoints in $\tilde{\Omega} \setminus \Omega$ such that any geodesic in \mathcal{A} is homotopic, within the set \mathcal{A} , to a geodesic lying outside Ω . Set of points lying on the geodesics in \mathcal{A} is denoted by $\Omega_{\mathcal{A}}$ i.e. $\Omega_{\mathcal{A}} = \bigcup_{\gamma \in \mathcal{A}} \gamma$ and $\partial_{\mathcal{A}}\Omega = \Omega_{\mathcal{A}} \cap \partial\Omega$. Now we will define what we mean by a geodesically convex subset.

Definition 2. *A subset K of the Riemannian manifold (Ω, g) is said to be geodesically convex if for any two points $x \in K$ and $y \in K$, the geodesic connecting them lies entirely in the set K .*

Finally, let $\mathcal{E}'(\tilde{\Omega})$ be the space of compactly supported tensor fields. We can then extend the definition of I by duality on tensor fields which are distributions in $\tilde{\Omega}$ supported in Ω , see [11]. Now we are ready to state the main theorems that we will prove in this article.

Theorem 2. *Let f be a symmetric m -tensor field on a simple real analytic manifold (Ω, g) with components in $\mathcal{E}'(\tilde{\Omega})$ and supported in Ω and K be a closed geodesically convex subset of Ω . If for each geodesic γ not intersecting K , we have that $I^0 f(\gamma) = 0$ then we can find a $(m-1)$ -tensor field v with components in $\mathcal{D}'(\text{int}(\tilde{\Omega}) \setminus K)$ such that $f = dv$ in $\text{int}(\tilde{\Omega}) \setminus K$ and $v = 0$ in $\text{int}(\tilde{\Omega}) \setminus \Omega$.*

Here we would like to mention that this theorem has been shown to be true for the case $m = 2$ in [11].

Theorem 3. *Let f be a symmetric m -tensor field on a simple real analytic manifold (Ω, g) with components in $\mathcal{E}'(\tilde{\Omega})$ and supported in Ω and K be a closed geodesically convex subset of Ω . If for each geodesic γ not intersecting K , we have that $I^q f(\gamma) = 0$ for $q = 0, 1, \dots, m$ then $\text{supp}(f) \subset K$.*

Theorem 4. *Let (Ω, g) be a simple real analytic manifold and g is real analytic in a neighborhood of $cl(\Omega)$. If for a symmetric m -tensor field f with components in $L^2(\Omega)$, we have that $I^q f = 0$ for $q = 0, 1, \dots, m$. Then $f = 0$.*

Here we would like to comment that the Theorem 4 also follows as a corollary of Theorem 3 when f is supported in Ω , however as we show in this paper that it can also be proved independently using s -injectivity of ray transform where we say $I^0 = I$ is s -injective if $If = 0$ implies $f^s = 0$. In the next section we will prove a proposition and some lemmas that will be needed for the proofs of our main theorems.

3 Preliminaries

We will now prove some results which are analogue of some results already proved for the case of symmetric 2-tensor fields in [11]. These will be needed later in the proof of our main theorems.

Fix a maximal geodesic γ_0 connecting $x_0 \neq y_0$ in the closure of $\tilde{\Omega}$. We construct normal coordinates $x = (x', x^n)$ at x_0 in $\tilde{\Omega}$ so that x^n is the distance to x_0 , and $\frac{\partial}{\partial x^n}$ is normal to $\frac{\partial}{\partial x^\alpha}$, $\alpha < n$, see [19, Section 2]. In these coordinates, the metric g satisfies $g_{ni} = \delta_{ni}$, for all i , and the Christoffel symbols satisfy $\Gamma_{nn}^i = \Gamma_{in}^n = 0$. Under these coordinates lines of the type $x' = \text{constant}$ are now geodesics with x^n as arc length parameter.

Let U be a tubular neighborhood of γ_0 in Ω , $U = \{(x', x^n) : |x'| < \epsilon, a(x') \leq x^n \leq b(x')\}$, where $\partial\Omega$ is locally given by $x^n = a(x')$ and $x^n = b(x')$. In the next proposition, we prove that for a symmetric m - tensor field f , one can always construct an $(m - 1)$ -tensor field v in U such that for

$$h := f - dv$$

one has

$$h_{i_1 \dots i_{m-1} n} = 0, \quad \text{for all possible values of } i_j \text{ and } v(x', a(x')) = 0.$$

\tilde{U} denotes the tubular neighborhood of γ_0 of the same type but in $\tilde{\Omega}$.

Remark 1. Numbers of n in the suffix of the tensor $v_{n \dots n i_1 \dots i_k}$ will be clear from the order of the tensor v . For example, if v is a m - tensor then

$$v_{n \dots n i_1 \dots i_k} = v_{\underbrace{n \dots n}_{m-k \text{ times}} i_1 \dots i_k}.$$

Proposition 1. *Let f be a symmetric m -tensor field then there exists a unique $(m-1)$ -tensor field such that for $h = f - dv$, we have*

$$h_{i_1 \dots i_{m-1} n} = 0, \quad \text{for all possible values of } i_j \text{ and } v(x', a(x')) = 0.$$

To prove this proposition, we need the following lemma for which we provide a proof in the appendix:

Lemma 1. *Let v be a symmetric $(m-1)$ -tensor field. Then for any $0 \leq k \leq m$, we have*

$$\begin{aligned} (dv)_{n\dots ni_k\dots i_1} &= \frac{(m-k)}{m} \frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^n} - \frac{2(m-k)}{m} \sum_{l=1}^k \Gamma_{i_l n}^p v_{n\dots ni_k\dots \hat{i}_l\dots i_1 p} \\ &\quad + \frac{1}{m} \sum_{l=1}^k \frac{\partial v_{n\dots ni_k\dots \hat{i}_l\dots i_1}}{\partial x^{i_l}} - \frac{2}{m} \sum_{l,q=1, l \neq q}^k \Gamma_{i_l i_q}^p v_{n\dots ni_k\dots \hat{i}_l\dots \hat{i}_q\dots i_1 p}. \end{aligned}$$

Now, let us come back to the proof of Proposition 1.

Proof of Proposition 1. Let us first recall the following definition:

$$(dv)_{i_1\dots i_m} = \sigma(i_1, \dots, i_m) \left(\frac{\partial v_{i_1\dots i_{m-1}}}{\partial x^{i_m}} - \sum_{l=1}^{m-1} \Gamma_{i_m i_l}^p v_{i_1, \dots, i_{l-1} p i_{l+1} \dots i_{m-1}} \right).$$

Proving

$$h_{i_1\dots i_{m-1} n} = 0$$

is equivalent to proving the existence of a $(m-1)$ -tensor field v such that

$$(dv)_{i_1\dots i_{m-1} n} = f_{i_1\dots i_{m-1} n}.$$

First we will prove the existence of a v such that

$$h_{n\dots n} = 0.$$

We will solve this equation together with the initial condition $v_{n\dots n}(x', a(x')) = 0$ to get $v_{n\dots n}$. After solving for $v_{n\dots n}$ we will consider

$$\begin{aligned} &h_{n\dots ni} = 0 \\ \Rightarrow &(dv)_{n\dots ni} = f_{n\dots ni} \\ \Rightarrow &\frac{\partial v_{n\dots ni}}{\partial x^n}(x) - 2\Gamma_{in}^p v_{n\dots np}(x) = \frac{m}{m-1} f_{n\dots ni}(x) - \frac{1}{m-1} \frac{\partial v_{n\dots n}}{\partial x^i}(x). \end{aligned}$$

Now we will solve this system of equations together with the initial conditions $v_{n\dots ni}(x', a(x')) = 0$ to get $v_{n\dots ni}$.

Proceeding in a similar manner let us assume that for a given k such that $0 \leq (k-1) \leq (m-1)$, we have already found $v_{n\dots ni_{k-1}\dots i_1}$ for which $h_{n\dots ni_{k-1}\dots i_1} = f_{n\dots ni_{k-1}\dots i_1} - (dv)_{n\dots ni_{k-1}\dots i_1} = 0$. If $(k-1) = (m-1)$ then we are done and if not then we can find $v_{n\dots ni_k\dots i_1}$ in the following manner. Using Lemma 1, we can construct the following system of equations for $h_{n\dots ni_k\dots i_1} = 0$.

$$\begin{aligned} \frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^n}(x) - 2 \sum_{l=1}^k \Gamma_{i_l n}^p v_{n\dots ni_k\dots \hat{i}_l\dots i_1 p}(x) &= \frac{1}{(m-k)} \left\{ m f_{n\dots ni_k\dots i_1}(x) - \sum_{l=1}^k \frac{\partial v_{n\dots ni_k\dots \hat{i}_l\dots i_1}}{\partial x^{i_l}}(x) \right. \\ &\quad \left. + 2 \sum_{l,q=1, l \neq q}^k \Gamma_{i_l i_q}^p v_{n\dots ni_k\dots \hat{i}_l\dots \hat{i}_q\dots i_1 p}(x) \right\}, \end{aligned}$$

Finally, we will solve the above system of equations with the initial conditions $v_{n\dots ni_k\dots i_1}(x', a(x')) = 0$ to get $v_{n\dots ni_k\dots i_1}$ uniquely. We repeat the same process till $k = (m - 1)$ to prove the proposition. \square

Lemma 2. *Let f be supported in Ω , and $I^0 f(\gamma) = 0$ for all maximal geodesics in \tilde{U} belonging to some neighborhood of the geodesics $x_0 = \text{const}$. Then $v = 0$ in $\text{int}(\tilde{U}) \setminus \Omega$.*

Proof. First let $f \in C^\infty(M)$. We will give another invariant definition of v and use it to conclude our lemma. For any $x \in \tilde{U}$ and any $\xi \in T_x \tilde{U} \setminus \{0\}$ so that $\gamma_{x,\xi}$ stays in \tilde{U} , we set

$$u(x, \xi) = \int_0^{l(x,\xi)} f_{i_1\dots i_m}(\gamma_{x,\xi}(t)) \dot{\gamma}_{x,\xi}^{i_1}(t) \cdots \dot{\gamma}_{x,\xi}^{i_m}(t) dt. \quad (1)$$

Extend the definition of $\gamma_{x,\xi}$ for $\xi \neq 0$ as a solution of the geodesic equation. Then $u(x, \xi)$ is homogeneous of order $(m - 1)$ in ξ . Consider

$$\begin{aligned} u(x, \lambda \xi) &= \lambda^{m-1} u(x, \xi) \\ \Rightarrow \quad \xi^{j_1} \cdots \xi^{j_{m-1}} \frac{\partial^{m-1}}{\partial \xi^{j_1} \cdots \partial \xi^{j_{m-1}}} u(x, \lambda \xi) &= (m-1)! u(x, \xi), \quad \text{diff. } (m-1) \text{ times w.r.t } \lambda \\ \Rightarrow \quad \xi^{j_1} \cdots \xi^{j_{m-1}} \frac{\partial^{m-1}}{\partial \xi^{j_1} \cdots \partial \xi^{j_{m-1}}} u(x, \xi) &= (m-1)! u(x, \xi), \quad \text{for } \lambda = 1. \end{aligned}$$

Now, we shall define a symmetric $(m - 1)$ - tensor field v as following:

$$v_{i_1\dots i_{m-1}}(x) = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \xi^{i_1} \cdots \partial \xi^{i_{m-1}}} u(x, \xi) \Big|_{\xi=e_n}. \quad (2)$$

Consider for any $0 \leq l \leq (m - 1)$

$$\begin{aligned} v_{i_1\dots i_{m-1-l}n\dots n}(x) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \xi^{i_1} \cdots \partial \xi^{i_{m-1-l}} \partial \xi^n \cdots \partial \xi^n} u(x, \xi) \Big|_{\xi=e_n} \\ &= \frac{1}{(m-1)!} \xi^{j_1} \cdots \xi^{j_l} \frac{\partial^{m-1}}{\partial \xi^{i_1} \cdots \partial \xi^{i_{m-1-l}} \partial \xi^{j_1} \cdots \partial \xi^{j_l}} u(x, \xi) \Big|_{\xi=e_n} \\ &= \frac{l!}{(m-1)!} \frac{\partial^{m-1}}{\partial \xi^{i_1} \cdots \partial \xi^{i_{m-1-l}}} u(x, \xi) \Big|_{\xi=e_n} \quad (\text{ using homogeneity of } u). \end{aligned}$$

Then, we have

$$v_{n\dots n}(x) = u(x, e_n).$$

We will now show that with this definition of v , for $h = f - dv$, one has

$$h_{i_1\dots i_{m-1}n} = 0, \quad \text{for all possible values of } i_j.$$

Define

$$w(x, \xi) = \int_0^{l(x,\xi)} h_{i_1\dots i_m}(\gamma_{x,\xi}(t)) \dot{\gamma}_{x,\xi}^{i_1}(t) \cdots \dot{\gamma}_{x,\xi}^{i_m}(t) dt. \quad (3)$$

Proposition 2. Let $0 \leq l \leq (m-1)$ and $w(x, \xi)$ is defined as above then

$$\left. \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} w(x, \xi) \right|_{\xi=e_n} = 0. \quad (4)$$

Proof. Consider for any $0 \leq l \leq (m-1)$,

$$\begin{aligned} & \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} w(x, \xi) \\ &= \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} u(x, \xi) - \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} \int_0^{l(x, \xi)} (dv)_{i_1 \dots i_m}(\gamma_{x, \xi}(t)) \dot{\gamma}_{x, \xi}^{i_1}(t) \dots \dot{\gamma}_{x, \xi}^{i_m}(t) dt \\ &= \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} u(x, \xi) - \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} \int_0^{l(x, \xi)} \frac{d}{dt} \left(v_{i_1 \dots i_{m-1}}(\gamma_{x, \xi}(t)) \dot{\gamma}_{x, \xi}^{i_1}(t) \dots \dot{\gamma}_{x, \xi}^{i_{m-1}}(t) \right) dt \\ &= \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} u(x, \xi) - \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} (v_{i_1 \dots i_{m-1}}(x) \xi^{i_1} \dots \xi^{i_{m-1}}) \\ &= \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} u(x, \xi) - \frac{(m-1)!}{l!} v_{j_1 \dots j_l n \dots n}(x) \\ \Rightarrow & \left. \frac{\partial^{m-1}}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} w(x, \xi) \right|_{\xi=e_n} = \left. \frac{\partial^l}{\partial \xi^{j_1} \dots \partial \xi^{j_l}} u(x, \xi) \right|_{\xi=e_n} - \frac{(m-1)!}{l!} v_{j_1 \dots j_l n \dots n}(x) \\ &= \frac{(m-1)!}{l!} v_{j_1 \dots j_l n \dots n}(x) - \frac{(m-1)!}{l!} v_{j_1 \dots j_l n \dots n}(x) \\ &= 0. \end{aligned}$$

□

Now let us recall the following relation [17, Section 1.2]

$$Gw(x, \xi) = h_{i_1 \dots i_m}(x) \xi^{i_1} \dots \xi^{i_m} \quad (5)$$

where $G = \xi^i \partial_{x^i} - \Gamma_{ij}^k \xi^i \xi^j \partial_{\xi^k}$ is the generator of the geodesic flow. After differentiating (5) $(m-1)$ times w.r.t. ξ , we get

$$\begin{aligned} & \frac{\partial^{m-1}}{\partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}} Gw(x, \xi) = m! h_{j_1 \dots j_{m-1} i}(x) \xi^i \\ \Rightarrow & \left. \frac{\partial^{m-1}}{\partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}} Gw(x, \xi) \right|_{\xi=e_n} = m! h_{j_1 \dots j_{m-1} n}(x). \end{aligned}$$

We will prove L.H.S. of the above equation is 0. This will prove our lemma. Consider

$$\begin{aligned}
\frac{\partial Gw(x, \xi)}{\partial \xi^{j_1}} &= \frac{\partial}{\partial \xi^{j_1}} \left(\xi^i \frac{\partial}{\partial x^i} w(x, \xi) \right) - \Gamma_{ij}^k \frac{\partial}{\partial \xi^{j_1}} \left(\xi^i \xi^j \frac{\partial}{\partial \xi^k} w(x, \xi) \right) \\
&= \frac{\partial w(x, \xi)}{\partial x^{j_1}} + \xi^i \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_1} \partial x^i} - \Gamma_{ij}^k \frac{\partial}{\partial \xi^{j_1}} (\xi^i \xi^j) \frac{\partial w(x, \xi)}{\partial \xi^k} - \Gamma_{ij}^k \xi^i \xi^j \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^k} \\
\Rightarrow \frac{\partial^2 Gw(x, \xi)}{\partial \xi^{j_1} \partial \xi^{j_2}} &= \frac{\partial^2 w(x, \xi)}{\partial x^{j_1} \partial \xi^{j_2}} + \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_1} \partial x^{j_2}} + \xi^i \frac{\partial^3 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^{j_2} \partial x^i} - \Gamma_{ij}^k \frac{\partial^2}{\partial \xi^{j_1} \partial \xi^{j_2}} (\xi^i \xi^j) \frac{\partial w(x, \xi)}{\partial \xi^k} \\
&\quad - \Gamma_{ij}^k \frac{\partial}{\partial \xi^{j_1}} (\xi^i \xi^j) \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_2} \partial \xi^k} - \Gamma_{ij}^k \frac{\partial}{\partial \xi^{j_2}} (\xi^i \xi^j) \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^k} - \Gamma_{ij}^k \xi^i \xi^j \frac{\partial^3 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^{j_2} \partial \xi^k} \\
&= \frac{\partial^2 w(x, \xi)}{\partial x^{j_1} \partial \xi^{j_2}} + \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_1} \partial x^{j_2}} + \xi^i \frac{\partial^3 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^{j_2} \partial x^i} - 2\Gamma_{j_1 j_2}^k \frac{\partial w(x, \xi)}{\partial \xi^k} \\
&\quad - 2\Gamma_{ij_1}^k \xi^i \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_2} \partial \xi^k} - 2\Gamma_{ij_2}^k \xi^i \frac{\partial^2 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^k} - \Gamma_{ij}^k \xi^i \xi^j \frac{\partial^3 w(x, \xi)}{\partial \xi^{j_1} \partial \xi^{j_2} \partial \xi^k}.
\end{aligned}$$

Using similar calculations, we get

$$\begin{aligned}
\frac{\partial^{m-1} Gw(x, \xi)}{\partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}} &= \xi^i \frac{\partial^m w(x, \xi)}{\partial \xi^{j_1} \dots \partial \xi^{j_{m-1}} \partial x^i} - \sum_{l, k=1, l \neq k}^{m-1} 2\Gamma_{j_1 j_l}^k \frac{\partial^{m-2} w(x, \xi)}{\partial \xi^k \partial \xi^{j_1} \dots \partial \xi^{\hat{j}_l} \dots \partial \xi^{\hat{j}_p} \dots \partial \xi^{j_{m-1}}} \\
&\quad + \sum_{l=1}^{m-1} \frac{\partial^{m-1} w(x, \xi)}{\partial x^{j_l} \partial \xi^{j_1} \dots \partial \xi^{\hat{j}_l} \dots \partial \xi^{j_{m-1}}} - \sum_{l=1}^{m-1} 2\Gamma_{ij_l}^k \xi^i \frac{\partial^{m-1} w(x, \xi)}{\partial \xi^k \partial \xi^{j_1} \dots \partial \xi^{\hat{j}_l} \dots \partial \xi^{j_{m-1}}} \\
&\quad - \Gamma_{ij}^k \xi^i \xi^j \frac{\partial^m w(x, \xi)}{\partial \xi^k \partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}}.
\end{aligned}$$

Which implies

$$\left. \frac{\partial^{m-1} Gw(x, \xi)}{\partial \xi^{j_1} \dots \partial \xi^{j_{m-1}}} \right|_{\xi=e_n} = 0, \quad (\text{Using Proposition 2 and } \Gamma_{nn}^k = 0).$$

Now that we have proved the proposition for the case when f is smooth, it can be extended to the case when f is a distribution by exactly the same reasoning as in [11, Lemma 3.1] \square

4 Proofs of Theorem 2 and Theorem 3

We will start with proving some lemmas and propositions required to prove our main theorems.

Lemma 3. *Let f be a symmetric m -tensor field as above. Let γ_0 be a geodesic of $\tilde{\Omega}$ and U be a neighborhood of γ_0 in $\tilde{\Omega}$. Assume that $WF_A(f) \cap \pi^{-1}(U)$ does not contain co-vectors of the type $(\xi', 0)$, then $h = f - dv$ also does not contain such co-vectors.*

Proof. Since v and dv have the same analytic wavefront set, so we will prove the lemma for v . We will prove this by induction by proving it for $v_{n...ni_k...i_1}$ for every $k \leq (m-1)$. Let us first do the analysis for $v_{n...n}$. Note that $v_{n...n}$ can be rewritten as a convolution with the Heaviside function in the following manner

$$\begin{aligned} v_{n...n}(x) &= \int_{-\infty}^{x^n} f_{n...n}(x', y^n) dy^n \\ &= \int_{-\infty}^{\infty} f_{n...n}(x', y^n) H(x^n - y^n) dy^n \end{aligned}$$

The wavefront set of the convolution can be found by applying [8, 8.2.16]. Since we have assumed that $\text{WF}_A(f) \cap \pi^{-1}(U)$ does not contain co-vectors of the type $(\xi', 0)$, hence it will be true for $v_{n...n}(x)$ as well. Now let us assume that the lemma holds for any $0 \leq k-1 < (m-1)$ i.e. $v_{n...ni_{k-1}...i_1}$ satisfies the same wavefront conditions. We will show that this implies that the Lemma 3 is true for k . For this consider the system of ODEs from Lemma 1,

$$\begin{aligned} \frac{\partial v_{n...ni_k...i_1}}{\partial x^n}(x) - 2 \sum_{l=1}^k \Gamma_{i_l n}^p v_{n...ni_k...i_l...i_1 p}(x) &= \frac{1}{(m-k)} \left\{ m f_{n...ni_k...i_1}(x) - \sum_{l=1}^k \frac{\partial v_{n...ni_k...i_l...i_1}}{\partial x^{i_l}}(x) \right. \\ &\quad \left. + 2 \sum_{l,q=1, l \neq q}^k \Gamma_{i_l i_q}^p v_{n...ni_k...i_l...i_q...i_1 p}(x) \right\}, \\ v_{n...ni_k...i_1}(x', a(x')) &= 0. \end{aligned}$$

This can be rewritten as :

$$\begin{aligned} \partial_n(\tilde{v}) - A(x', x^n) \tilde{v} &= w, \\ \tilde{v}|_{x^n < 0} &= 0 \end{aligned}$$

where A is an analytic matrix, $\tilde{v} = v_{n...ni_k...i_1}$ and $\text{WF}_A(w) \cap \pi^{-1}(U)$ does not have covectors of the type $(\xi', 0)$. By Duhamel's principle the solution to the above is given by :

$$\tilde{v}(x', x^n) = \int_{-\infty}^{x^n} \Phi(x', x^n, y^n) w(x', y^n) dy^n$$

where Φ is analytic. The expression given above for $\tilde{v}(x', x^n)$ can be rewritten as :

$$\tilde{v}(x', x^n) = \int_{R^n} \Phi(x', x^n, y^n) H(x^n - y^n) \delta(x' - y') w(y', y^n) dy' dy^n$$

The kernel of the integral operator is given by : $\Phi(x', x^n, y^n) H(x^n - y^n) \delta(x' - y')$. Note that the frequency set of the analytic wavefront set of the Heaviside and Delta distributions here are perpendicular to each other and hence satisfy Hörmander's non cancellation condition [8, 8.5.3]. The lemma then follows from the argument in [11]. \square

4.1 Analyticity along Conormal Directions

Before moving further, we will need the following Proposition which is an analogue of Proposition 2 from [21] and generalizes that proposition for the case when f is a symmetric m -tensor. We will mimic the proof for the case when $m = 2$ as given in that paper and adapt the arguments wherever necessary to make it work for a symmetric tensor field of any order.

Proposition 3. *Let Ω and f be as above. Let γ_0 be a fixed geodesic through x_0 normal to ξ_0 where $(x_0, \xi_0) \in T^*\Omega \setminus 0$. Assume $(I^0 f)(\gamma) = 0$ for all γ in a neighborhood of γ_0 and g is analytic in this neighborhood. Let $\delta f = 0$ near x_0 . Then*

$$(x_0, \xi_0) \notin WF_A(f).$$

Proof. For the given geodesic γ_0 that passes through x_0 and is normal to ξ_0 , let us consider a tubular neighborhood U of γ_0 endowed with analytic semi-geodesic coordinates $x = (x', x^n)$ on it. Without loss of generality, assume that $x_0 = 0$. Furthermore, $\forall x \in \gamma_0, x' = 0$. Note that $U = \{(x', x^n) : |x'| < \epsilon \text{ and } l^- < x^n < l^+; 0 < \epsilon \ll 1\}$ in this co-ordinate system. Choose ϵ such that $\{x : x^n = l^-, l^+ \text{ and } |x'| < \epsilon\}$ lies outside Ω . Clearly $\xi_0 = (\xi'_0, 0)$. Hence our goal is now to show:

$$(0, \xi_0) \notin WF_A(f).$$

As stated earlier, we will reproduce the arguments from [21] here for the sake of completeness. Consider $Z = \{|x| < \frac{7\epsilon}{8} : |x_n| = 0\}$ and let x' variable be denoted on Z by z' . Then (z', θ') are local co-ordinates in $\text{nbd}(\gamma_0)$ given by $(z', \theta') \rightarrow \gamma_{(z', 0), (\theta', 1)}$. Here, $|\theta'| \ll 1$ (where, the geodesic is in the direction $(\theta', 1)$). By following their arguments verbatim, we get the following equation :

$$\int e^{i\lambda z'(x, \theta') \cdot \xi'} a_N(x, \theta') f_{i_1 \dots i_m}(x) b^{i_1}(x, \theta') \dots b^{i_m}(x, \theta') dx = 0 \quad (6)$$

Here, $(x, \theta') \rightarrow a_N$ is analytic and satisfies

$$|\partial^\alpha a_N| \leq (CN)^{|\alpha|}, \quad \alpha \leq N, \quad (7)$$

see [21, equation (38)]. Also, note that $b(0, \theta') = \theta$ and $a_N(0, \theta') = 1$.

Further, let us choose $\theta(\xi)$ to be a vector depending analytically on ξ near $\xi = \xi_0$ and satisfying the following conditions :

$$\begin{aligned} \theta(\xi) \cdot \xi &= 0, \quad \theta^n(\xi) = 1 \quad \text{and} \\ \theta(\xi_0) &= (0, \dots, 1) = e_n \end{aligned}$$

Now, we will rewrite (6) using the above mapping in the following form:

$$\int e^{i\lambda \phi(x, \xi)} \tilde{a}_N(x, \xi) f_{i_1 \dots i_m}(x) \tilde{b}^{i_1}(x, \xi) \dots \tilde{b}^{i_m}(x, \xi) dx = 0. \quad (8)$$

Here $\phi(x, \xi) = z' \cdot \xi'$. This phase function has been shown in [21] to be non-degenerate in a neighborhood of $(0, \xi_0)$ by showing $\phi_{x\xi}(0, \xi) = \text{Id}$. This also implies that $x \rightarrow \phi_\xi(x, \xi)$ is a diffeomorphism in this neighborhood.

To establish the above condition in a neighborhood of the geodesic γ_0 , one chooses the co-normal vector

$$\xi_0 = e_{n-1}, \quad \text{i.e. the covector } (0, 0, \dots, 0, 1, 0) \quad (9)$$

and defines

$$\theta(\xi) = (\xi_1, \dots, \xi_{n-2}, -\frac{\xi_1^2 + \dots + \xi_{n-2}^2 + \xi_n}{\xi_{n-1}}, 1).$$

This definition of θ is consistent with the requirement put on $\theta(\xi)$ as above. One can then show that the differential of the map $\xi \rightarrow \theta(\xi)$ where $\xi \in S^{n-1}$ is invertible at $\xi_0 = e_{n-1}$, see [21, equation (44)].

Lemma 4. [21, Lemma 3.2] *Let, $\theta(\xi)$ and $\phi(x, \xi)$ be as above. Then, $\exists \delta > 0$ such that if*

$$\phi_\xi(x, \xi) = \phi_\xi(y, \xi)$$

for some $x \in U$, $|y| < \delta$, $|\xi - \xi_0| < \delta$ where ξ is complex, then $y = x$.

We will study the analytic wavefront set of f using Sjöstrand's complex stationary phase method. For this assume x, y as in Lemma 4 and $|\xi_0 - \eta| < \frac{\delta}{C}$ with $\tilde{C} \gg 2$ and $\delta \ll 1$. Multiply (8) by

$$\tilde{\chi}(\xi - \eta) e^{i\lambda \left(i \frac{(\xi - \eta)^2}{2} - \phi(y, \xi) \right)}$$

where $\tilde{\chi}$ is the characteristic function of the ball $B(0, \delta) \subset \mathbb{C}^n$ and then integrate w.r.t. ξ to get :

$$\iint e^{i\lambda \Phi(y, x, \xi, \eta)} \tilde{a}_N(x, \xi) f_{i_1 \dots i_m}(z) \tilde{b}^{i_1}(x, \xi) \dots \tilde{b}^{i_m}(x, \xi) dx d\xi = 0. \quad (10)$$

In the above equation, $\tilde{a}_N = \tilde{\chi}(\xi - \eta) \tilde{a}_N$ is another analytic and elliptic amplitude for x close to zero and $|\xi - \eta| < \frac{\delta}{C}$ and

$$\Phi = -\phi(y, \xi) + \phi(x, \xi) + \frac{i}{2}(\xi - \eta)^2.$$

Furthermore,

$$\Phi_\xi = \phi_\xi(x, \xi) - \phi_\xi(y, \xi) + i(\xi - \eta).$$

To apply the stationary phase method we need to know the critical points of $\xi \mapsto \Phi$. Using the Lemma 4 above we have :

1. If $y = x$, \exists a unique real critical point $\xi_c = \eta$
2. If $y \neq x$, there are no real critical points

3. Also by Lemma 4, if $y \neq x$, there is a unique complex critical point if $|x - y| < \delta/C_1$ and no critical points for $|x - y| > \delta/C_0$ for some constants C_0 and C_1 with $C_1 > C_0$.

Define, $\psi(x, y, \eta) := \Phi(\xi_c)$. Then at $x = y$

$$(i) \psi_y(x, x, \eta) = -\phi_x(x, \eta) \quad (ii) \psi_x(x, x, \eta) = \phi_x(x, \eta) \quad (iii) \psi(x, x, \eta) = 0.$$

Now, we split the x integral in (10) in to two parts : we integrate over $\{x : |x - y| > \delta/C_0\}$ for some $C_0 > 1$ and its complement. Since, $|\Phi_\xi|$ has a positive lower bound for $\{x : |x - y| > \delta/C_0\}$ and there are no critical points of $\xi \rightarrow \Phi$ in this set, we can estimate that integral in the following manner: First note that, $e^{i\lambda\Phi(x, \xi)} = \frac{\Phi_\xi \partial_\xi}{i\lambda|\Phi_\xi|^2} e^{i\lambda\Phi(x, \xi)}$. Using, (7) and integrating by parts N times with respect to ξ and the fact that on the boundary $|\xi - \eta| = \delta$, we get

$$\left| \iint_{|x-y|>\delta/C_0} e^{i\lambda\Phi(y, x, \xi, \eta)} \tilde{a}_N(x, \xi) f_{i_1 \dots i_m}(x) \tilde{b}^{i_1}(x, \xi) \dots \tilde{b}^{i_m}(x, \xi) dx d\xi \right| \leq C \left(\frac{CN}{\lambda} \right)^N + CN e^{-\frac{\lambda}{C}} \quad (11)$$

We choose $N \leq \lambda/Ce \leq N + 1$ to get an exponential error on the right. Now in estimating the integral

$$\left| \int_{|x-y|\leq\delta/C_0} e^{i\lambda\Phi(y, x, \xi, \eta)} \tilde{a}_N(x, \xi) f_{i_1 \dots i_m}(x) \tilde{b}^{i_1}(x, \xi) \dots \tilde{b}^{i_m}(x, \xi) dx d\xi \right| \quad (12)$$

we use [18, Theorem 2.8] and the remark following that to conclude:

$$\int_{|x-y|\leq\delta/C_0} e^{i\lambda\psi(x, \alpha)} f_{i_1 \dots i_m}(x) B^{i_1 \dots i_m}(x, \alpha; \lambda) dx = \mathcal{O}(e^{-\lambda/C}) \quad (13)$$

where $\alpha = (y, \eta)$ and B is a classical analytical symbol with principal part $\tilde{b} \otimes \dots \otimes \tilde{b}$. See appendix below for a proof of estimates in (11) and (13)

Let, $\beta = (y, \mu)$ where, $\mu = \phi_y(y, \eta) = \eta + \mathcal{O}(\delta)$. At $y = 0$, we have $\mu = \eta$. Also $\alpha \rightarrow \beta$ is a diffeomorphism following similar analysis as in [21, Section 4]. If we write $\alpha = \alpha(\beta)$, then the above equation becomes:

$$\int_{|x-y|\leq\delta/C_0} e^{i\lambda\psi(x, \beta)} f_{i_1 \dots i_m}(x) B^{i_1 \dots i_m}(x, \beta; \lambda) dx = \mathcal{O}(e^{-\lambda/C}) \quad (14)$$

where ψ satisfies (i), (ii) and (iii), and B is a classical analytical symbol as before and :

$$\psi_y(x, x, \eta) = -\mu, \quad \psi_x(x, x, \eta) = \mu \quad \text{and} \quad \psi_y(x, x, \eta) = 0$$

The symbols in (14) satisfy :

$$\sigma_P(B)(0, 0, \mu) = \theta(\mu) \otimes \dots \otimes \theta(\mu) = \theta^{\otimes m}(\mu)$$

and in particular,

$$\sigma_P(B)(0, 0, \xi_0) = e_n \otimes \dots \otimes e_n.$$

Let, $\theta_1 = e_n, \theta_2, \dots, \theta_N$ be $N = \binom{n+m-2}{m}$ unit vectors lie in the hyperplane perpendicular to ξ_0 . We will also assume that $\{\theta_i^{\odot m}\}_{i=1}^N$ are independent, where \odot is a symmetrized product of vectors. Existence of such vectors in any open set in ξ_0^\perp can be shown. We can therefore assume that θ_p belongs in a small neighborhood around $\theta_1 = e^n$. Then we can rotate the axes a little such that $\xi_0 = e^{n-1}$ and $\theta_p = e_n$ and do the same construction as above. This gives us $N = \binom{n+m-2}{m}$ phase functions $\psi_{(p)}$, and as many number of analytic symbols for which (14) is true i.e.

$$\int_{|x-y| \leq \delta/C_0} e^{i\lambda\psi_{(p)}(x,\beta)} f_{i_1 \dots i_m}(x) B_{(p)}^{i_1 \dots i_m}(x, \beta; \lambda) dx = \mathcal{O}(e^{-\lambda/C}) \quad (15)$$

where

$$\sigma_P(B_p)(0, 0, \mu) = \theta_p(\mu) \otimes \dots \otimes \theta_p(\mu), \quad p = 1, \dots, N \quad \text{up to elliptic factors.}$$

Now we use the fact that $\delta f = 0$ near x_0 . So integrating

$$\frac{1}{\lambda} \exp(i\lambda\psi_{(1)}(x, \beta)) \chi_0 \delta f = 0$$

w.r.t. x and after an integration by parts, we get

$$\int_{|x-y| \leq \delta/C_0} e^{i\lambda\psi_{(1)}(x,\beta)} f_{i_1 \dots i_m}(x) C^{i_m}(x, \beta; \lambda) dx = \mathcal{O}(e^{-\lambda/C}), \quad i_j \in \{1, \dots, n\} \text{ and } j = 1, \dots, (m-1) \quad (16)$$

for $\beta_x = y$ small enough and where $\sigma_P(C^{i_m})(0, 0, \xi_0) = (\xi_0)^{i_m}$. This gives us additional $\tilde{N} = \binom{n+m-2}{m-1}$ equations such that the system of $N + \tilde{N} = \binom{n+m-1}{m}$ equations (15), (16) can be viewed as a tensor valued operator on f . We claim that the symbol for this operator is elliptic at $(0, 0, \xi_0)$. Indeed, to show that the symbol is elliptic at $(0, 0, \xi_0)$ amounts to showing that the only solution to following system of equations is $f = 0$:

$$\theta_p^{i_1} \dots \theta_p^{i_m} f_{i_1 \dots i_m} = 0, \quad \text{for all } p = \{1 \dots N\} \quad (17)$$

$$\xi_0^{i_m} f_{i_1 \dots i_m} = 0, \quad \text{for } 1 \leq i_1 \leq \dots \leq i_{m-1} \leq n \quad (18)$$

Using conditions on θ_p and ξ_0 , it is proved in [10] that above system of equations will imply $f = 0$. \square

For the more general case, when δf is microlocally analytic at (x_0, ξ_0) , we use the same arguments as above, except that we multiply (14) by an appropriate cut-off near (x_0, x_0, ξ_0) and use integration by parts as explained in [11, Section 4] to conclude the following proposition:

Proposition 4. *Let $\tilde{\Omega}$, f and γ_0 be as in the statement of Proposition 3. If $(x_0, \xi_0) \notin WF_A(\delta f)$ (where ξ_0 is normal to the geodesic γ_0 at x_0), and $I^0 f(\gamma) = 0$ for all γ in a nbd. of γ_0 , then $(x_0, \xi_0) \notin WF_A(f)$.*

The rest of the argument from [11] applies as it is and thereby we prove Theorem 2. We will briefly outline the ideas here for the sake of completeness: We will first need to show that the following analogue of [11, Theorem 2.2(a)] holds for the case of symmetric m tensor fields as well :

Theorem 5. *Let f be as above. Then $I^0 f(\gamma) = 0$ for each geodesic γ in \mathcal{A} , if and only if for each geodesic $\gamma_0 \in \mathcal{A}$ there exists a neighborhood \mathcal{U} of γ_0 and a $(m-1)$ -tensor field $v \in \mathcal{D}'(\widetilde{\Omega}_{\mathcal{U}})$ such that $f = dv$ in $\widetilde{\Omega}_{\mathcal{U}}$, and $v = 0$ outside Ω .*

The “if” part follows from the Fundamental Theorem of Calculus. To prove the “only if” part of the theorem assume that γ_0 is a geodesic in the set \mathcal{A} , where \mathcal{A} is defined in Section 3. This means that it can be continuously deformed within the set to a point. Hence by extending all geodesics in Ω to maximal geodesics in $\tilde{\Omega}$, we know that there must exist two continuous curves $a(t), b(t), t \in [0, 1]$ such that $\gamma_{(a(0), b(0))}$ is tangent to $\partial\Omega$, $\gamma_{(a(t), b(t))} \in \mathcal{A}$ and $\gamma_{(a(1), b(1))}$ is γ_0 . Using [12, Theorem A], one can show that the Theorem 5 is at least true in a small neighborhood of $\partial\Omega$ i.e. in some neighborhood of the geodesics $\gamma_{(a(t), b(t))}$ for $0 \leq t \leq 2t_0$ for some $t_0 \ll 1$. More precisely,

Lemma 5. *[11, Lemma 5.1] There exists a neighborhood V of $\partial\Omega$ such that $\forall x \in V$, $\text{dist}(x, \partial\Omega) < \epsilon_0$ for some $\epsilon_0 > 0$ and a unique v_0 such that $f = dv_0$ in V , $v_0 = 0$ on $\partial\Omega$ and v_0 is analytic in V , up to the boundary $\partial\Omega$.*

Note that the above implies that in V , the tensor $h = f - dv$ as constructed in Proposition 1 is zero. We will now construct a sequence of neighborhoods beginning with a neighborhood of $\gamma_{(a(0), b(0))}$ and up to a neighborhood of $\gamma_{(a(1), b(1))}$ for which the locally defined tensor field $h = f - dv$ is zero. However to implement this program we will need the following theorem due to Sato-Kawai-Kashiwara, see e.g. [14] or [23] :

Lemma 6. *[23, Lemma 3.1] Let $f \in \mathcal{D}'(\Omega)$. Let $x_0 \in \Omega$ and let U be a neighborhood of x_0 . Assume that S is a C^2 submanifold of Ω and $x_0 \in \text{supp}(f) \cap S$. Furthermore, let S divide U into two open connected sets and assume that $f = 0$ on one of these open sets. Let $\xi \in N_{x_0}^*(S) \setminus 0$, then $(x_0, \xi) \in WF_A(f)$*

Consider the cone of all vectors in $T_{a(t)}\tilde{\Omega}$ at an angle less than ϵ with $\dot{\gamma}_{[a(t), b(t)]}$ for some small properly chosen ϵ . The cone $C_\epsilon(t)$ with its vertex at $a(t) \in \partial\tilde{\Omega}$ is then the image of the above cone of vectors under the exponential map. We will choose $\epsilon > 0$ such that

1. $C_{2\epsilon}(t) \subset \tilde{\Omega}_{\mathcal{A}}, \forall t \in [0, 1]$
2. $C_\epsilon(t) \subset \tilde{V}$ for $0 \leq t \leq t_0$ where $\tilde{V} := V \cup (\tilde{\Omega}/\Omega)$.
3. No geodesic inside the cone $cl(C_{2\epsilon}(t))$, $t_0 < t < 1$, with vertex at $a(t)$ is tangent to $\partial\Omega$.

For any t , let us construct a tensor field h_t in $C_\epsilon(2t)$ just as in Proposition 1. Recall that the support of h_t lies in Ω . Since $C_\epsilon(t) \subset \tilde{V}$ for $0 \leq t \leq t_0$ then by Lemma 5 we have

$h_t = 0$ in $C_\epsilon(t) \subset \tilde{V}$. Hence the set $\{t \in [0, 1] : h_t = 0 \text{ in } C_\epsilon(t)\}$ is non empty. Let $t^* = \sup\{t \in [0, 1] : h_t = 0 \text{ in } C_\epsilon(t)\}$. We will show: $t^* = 1$. This will imply that there exists a neighborhood \mathcal{U} of γ_0 and a $(m-1)$ tensor field $v \in \mathcal{D}'(\widetilde{\Omega_{\mathcal{U}}})$ such that $h = f - dv = 0$ there.

Assume $t^* < 1$. Then $h_{t^*} = 0$ in $C_\epsilon(t^*)$ because $h_{t^*} = 0$ outside Ω . Next we will show that $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$. This gives us a contradiction, because on increasing t^* slightly to t , we can get $C_\epsilon(t) \cap \Omega \subset C_{2\epsilon}(t^*) \cap \Omega$ such that h_t is zero in this $C_\epsilon(t)$. Here we would like to mention that as h_t is got by solving an Initial Value Problem for a system of ODEs, hence they are locally unique. In particular, if $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$ and $C_\epsilon(t) \cap \Omega \subset C_{2\epsilon}(t^*) \cap \Omega$, then $h_t = 0$ in $C_\epsilon(t)$ which contradicts the choice for t^* . To fulfill our program, consider h_{t^*} in $C_{2\epsilon}(t^*)$. As stated earlier, $h_{t^*} = 0$ in $C_\epsilon(t^*)$. Let $\epsilon < \epsilon_0 \leq 2\epsilon$ be such that $C_{\epsilon_0}(t^*)$ is the first cone whose boundary intersects $\text{supp}(h_{t^*})$. If no such ϵ_0 can be found then we are done. Let $q \in \text{supp}(h_{t^*}) \cap \partial C_{\epsilon_0}(t)$. Clearly $q \notin \partial\Omega$, because $h_{t^*} = 0$ outside Ω . So q is an interior point of $\tilde{\Omega}$. In $\tilde{\Omega}$, $(\delta f)_{i_1 \dots i_{m-1}} = (\delta(f\chi))_{i_1 \dots i_{m-1}}$ where χ is the characteristic function of Ω . But one can first prove the theorem for f such that $f = f^s$ in Ω and then make the argument for any general f . Working first with such tensor fields for which $f = f^s$, one knows that such a tensor field is analytic in $\partial\Omega$ up to $\partial\Omega$, see [11, Section 5].

Now,

$$\begin{aligned} (\delta(f^s\chi))_{i_1 \dots i_{m-1}} &= (\nabla_k(f_{i_1 \dots i_{m-1}j}^s\chi))g^{jk} \\ &= (\chi\nabla_k f_{i_1 \dots i_{m-1}j}^s)g^{jk} + f_{i_1 \dots i_{m-1}j}^s g^{jk} \nabla_k \chi \\ &= f_{i_1 \dots i_{m-1}j}^s \nabla^j \chi \\ &= -f_{i_1 \dots i_{m-1}j}^s \nu^j \delta_{\partial\Omega}. \end{aligned}$$

This shows that the analytic wavefront set of δf is in $N^*(\partial\Omega)$. Let $\tilde{\gamma}$ be the geodesic in Ω on the surface of $\partial C_{\epsilon_0}(t^*)$ that contains q . Because $N^*\tilde{\gamma}$ does not intersect $N^*\partial\Omega$, by Proposition 4 and by Lemma 3, h has no analytic singularities in $N^*\tilde{\gamma}$. Consider a small open set W containing q which is divided by the surface of $\partial C_{\epsilon_0}(t^*)$ into two open connected sets as in the statement of Lemma 6 and $h_{t^*} = 0$ in one of these open sets. Since the co-normals to $C_{\epsilon_0}(t^*)$ at q are not in $WF_A(h_{t^*})$, this implies $q \notin \text{supp}(h_{t^*})$ by the Sato-Kawai-Kashiwara theorem mentioned above. This shows that $h_{t^*} = 0$ in $C_{2\epsilon}(t^*)$ which in turn implies $t^* = 1$. This proves Proposition 5.

Using the condition that any closed path with a base point on $\partial\Omega$ is homotopic to a point lying on $\partial\Omega$ and using the geometric arguments in Section 6 of [11] along with Proposition 5, we conclude the proof of Theorem 2.

Remark: The symmetric $m-1$ tensor field v also has components in $\mathcal{E}'(\tilde{\Omega})$ and is supported in Ω just like the m - tensor field f .

4.2 Proof of Theorem 3

Proof of Theorem 3. We will first prove the following lemma :

Lemma 7. *For any $1 \leq k \leq m$, if $f = dv$ with $v|_{\partial\Omega} = 0$. Then $I^k f = -kI^{k-1}v$.*

Proof. Consider

$$\begin{aligned}
I^k f(\gamma) &= I^k(dv)(\gamma) \\
&= \int_0^{l(\gamma)} t^k (dv)_{i_1 \dots i_m}(\gamma(t)) \dot{\gamma}^{i_1}(t) \dots \dot{\gamma}^{i_m}(t) dt \\
&= \int_0^{l(\gamma)} t^k \frac{d}{dt} \{v_{i_1 \dots i_{m-1}}(\gamma(t)) \dot{\gamma}^{i_1}(t) \dots \dot{\gamma}^{i_{m-1}}(t)\} dt \\
&= \{t^k v_{i_1 \dots i_{m-1}}(\gamma(t)) \dot{\gamma}^{i_1}(t) \dots \dot{\gamma}^{i_{m-1}}(t)\}_0^{l(\gamma)} \\
&\quad - k \int_0^{l(\gamma)} t^{k-1} v_{i_1 \dots i_{m-1}}(\gamma(t)) \dot{\gamma}^{i_1}(t) \dots \dot{\gamma}^{i_{m-1}}(t) dt \\
&= -k I^{k-1} v(\gamma),
\end{aligned}$$

where first term in the second last equality is 0 because of our assumption $v|_{\partial\Omega} = 0$. Thus, we have our lemma. \square

Let us come back to the proof of Theorem 3. As we know from Theorem 2 that if $I^0 f(\gamma) = If(\gamma) = 0$ for each geodesic γ not intersecting K then there exist $(m-1)$ -tensor field v_1 which is 0 on the boundary $\partial\Omega$ such that $f = dv_1$ on $\Omega \setminus K$. And from the Lemma 7, we know

$$I^1 f(\gamma) = I^1(dv_1)(\gamma) = -I^0 v_1(\gamma).$$

Again using Theorem 2 we conclude that there exist $(m-2)$ -tensor field v_2 such that $v_1 = dv_2$ and $v_2|_{\partial\Omega} = 0$. Using Theorem 2 along with Lemma 7 $(m-2)$ more times, we have

$$I^m f(\gamma) = m!(-1)^m I^0 v_m(\gamma) = 0$$

where v_m is 0-tensor i.e. a function. Now using [9, Theorem 1], we can conclude $v_m = 0$ on $\Omega \setminus K$. And since $f = d^m v_m$ on an open connected set $\Omega \setminus K$ therefore f is also 0 on $\Omega \setminus K$. \square

5 Proof of Theorem 4

To prove Theorem 4, we will need the s -injectivity of the ray transform for symmetric m -tensor fields. The proof of s -injectivity for symmetric 2-tensor fields is given in [20]. The same proof will also work for a symmetric tensor field of any order. For details, we will refer the reader to [20, Sections 2,3,4]. Hence we have :

Theorem 6. *[20, Theorem 1.4] Let (Ω, g) be a compact simple real analytic manifold with smooth boundary and f be a symmetric m -tensor field with components in $L^2(\Omega)$. If $I^0 f(\gamma) = 0$ for all γ which are geodesics in Ω , then $f^s = 0$ in Ω .*

Theorem 7. Let Ω be a compact simple Riemannian manifold with boundary. Let $m \geq 0$ and $p \geq m$ be integers. Then for any $f \in L^2(S^m(\Omega))$, there exist uniquely determined v_0, \dots, v_m with $v_i \in H^i(S^{m-i}\Omega)$ for $i = 0, 1, \dots, m$ such that

$$f = \sum_{i=0}^m d^i v_i, \quad \text{with } v_i \text{ solenoidal for } 0 \leq i \leq m-1$$

$$\text{and for each } 0 \leq i \leq m-1, \quad \sum_{j=0}^i d^j v_{m-i+j} = 0 \text{ on } \partial\Omega.$$

Proof. This follows from a repeated application of [17, Theorem 3.3.2]. \square

Proof of Theorem 4. We have from Theorem 7 that

$$\begin{aligned} f &= \sum_{i=0}^m d^i v_i, \quad \text{with } v_i \text{ solenoidal for } 0 \leq i \leq m-1 \\ \text{and for each } 0 \leq i \leq m-1, \quad &\sum_{j=0}^i d^j v_{m-i+j} = 0 \text{ on } \partial\Omega. \end{aligned} \quad (19)$$

Using s -injectivity of I , we know that $v_0 = 0$, since it is solenoidal. Now consider

$$\begin{aligned} 0 &= I^1 f(\gamma) = I^1 \left(\sum_{i=0}^m d^i v_i \right) (\gamma) \\ &= I^1 \left(d \left(\sum_{i=1}^m d^{i-1} v_i \right) \right) (\gamma), \quad \text{since } v_0 = 0 \\ &= -I^0 \left(\sum_{i=1}^m d^{i-1} v_i \right) (\gamma) \quad (\text{using Lemma 7}) \end{aligned}$$

From this, we can conclude v_1 is also 0 because it is solenoidal part of tensor field $\sum_{i=1}^m d^{i-1} v_i$.

Now suppose that v_1, \dots, v_k can be shown to be equal to 0 from the knowledge of $I^1 f, \dots, I^k f$. Then

$$\begin{aligned} 0 &= I^{k+1} \left(f - \sum_{i=0}^k d^i v_i \right) = I^{k+1} \left(\sum_{i=k+1}^m d^i v_i \right) \\ \Rightarrow \quad &I^{k+1} \left(\sum_{i=k+1}^m d^i v_i \right) = 0 \\ \Rightarrow \quad &(-1)^{k+1} (k+1)! I^0 \left(\sum_{i=k+1}^m d^{i-k-1} v_i \right) = 0, \quad (\text{using Lemma 7, } (k+1) \text{ times}). \end{aligned}$$

Therefore $v_{k+1} = 0$ because it is the solenoidal part of the tensor field $(\sum_{i=k+1}^m d^{i-k-1} v_i)$. By induction, the proof is now complete. \square

6 Appendix

Proof of Lemma 1. First, let us recall for a $(m-1)$ -tensor field v ,

$$(\mathrm{d}v)_{i_1 \dots i_m} = \sigma(i_1, \dots, i_m) \left(\frac{\partial v_{i_1 \dots i_{m-1}}}{\partial x^{i_m}} - \sum_{l=1}^{m-1} \Gamma_{i_m i_l}^p v_{i_1, \dots, i_{l-1} p i_{l+1} \dots i_{m-1}} \right).$$

We will prove this result for $k = 0, 1, 2$ and then for general $k \leq m$.

$$\begin{aligned} (\mathrm{d}v)_{n \dots n} &= \frac{\partial v_{n \dots n}}{\partial x^n}, \quad \text{for } k = 0 \\ (\mathrm{d}v)_{n \dots ni} &= \frac{m-1}{m} \frac{\partial v_{n \dots ni}}{\partial x^n} - \frac{2(m-1)}{m} \Gamma_{in}^p v_{n \dots np} + \frac{1}{m} \frac{\partial v_{n \dots n}}{\partial x^i}, \quad \text{for } k = 1 \end{aligned}$$

And for $k = 2$, we have

$$\begin{aligned} (\mathrm{d}v)_{n \dots nij} &= \sigma(n, \dots, n, i, j) \left(\frac{\partial v_{n \dots ni}}{\partial x^j} - \Gamma_{ij}^p v_{n \dots np} - (m-2) \Gamma_{nj}^p v_{n \dots nip} \right) \\ &= \frac{\sigma(n, \dots, n, i)}{m} \left\{ (m-1) \frac{\partial v_{n \dots nij}}{\partial x^n} + \frac{\partial v_{n \dots ni}}{\partial x^j} - 2 \Gamma_{ij}^p v_{n \dots np} - (m-2) \Gamma_{in}^p v_{n \dots npj} \right. \\ &\quad \left. - 2(m-2) \Gamma_{nj}^p v_{n \dots nip} - (m-2)^2 \Gamma_{in}^p v_{n \dots npj} \right\} \\ &= \frac{\sigma(n, \dots, n, i)}{m} \left\{ (m-1) \frac{\partial v_{n \dots nij}}{\partial x^n} + \frac{\partial v_{n \dots ni}}{\partial x^j} - 2 \Gamma_{ij}^p v_{n \dots np} - 2(m-2) \Gamma_{nj}^p v_{n \dots nip} \right. \\ &\quad \left. - (m-1)(m-2) \Gamma_{in}^p v_{n \dots npj} \right\} \\ &= \frac{1}{m(m-1)} \left\{ (m-1) \left((m-2) \frac{\partial v_{n \dots nij}}{\partial x^n} + \frac{\partial v_{n \dots nj}}{\partial x^i} \right) + (m-1) \frac{\partial v_{n \dots ni}}{\partial x^j} \right. \\ &\quad \left. - 2(\Gamma_{ij}^p v_{n \dots np} + (m-2) \Gamma_{nj}^p v_{n \dots npi}) - 2(m-2) (\Gamma_{ij}^p v_{n \dots np} + (m-2) \Gamma_{nj}^p v_{n \dots npi}) \right. \\ &\quad \left. - 2(m-1)(m-2) \Gamma_{in}^p v_{n \dots npj} \right\} \\ &= \frac{1}{m(m-1)} \left\{ (m-1)(m-2) \frac{\partial v_{n \dots nij}}{\partial x^n} + (m-1) \frac{\partial v_{n \dots nj}}{\partial x^i} + (m-1) \frac{\partial v_{n \dots ni}}{\partial x^j} \right. \\ &\quad \left. - 2(m-1) \Gamma_{ij}^p v_{n \dots np} - 2(m-1)(m-2) \Gamma_{nj}^p v_{n \dots npi} - 2(m-1)(m-2) \Gamma_{in}^p v_{n \dots npj} \right\} \\ &= \frac{1}{m} \left\{ (m-2) \frac{\partial v_{n \dots nij}}{\partial x^n} + \frac{\partial v_{n \dots nj}}{\partial x^i} + \frac{\partial v_{n \dots ni}}{\partial x^j} - 2(m-2) \Gamma_{nj}^p v_{n \dots npi} - 2(m-2) \Gamma_{in}^p v_{n \dots npj} \right. \\ &\quad \left. - 2 \Gamma_{ij}^p v_{n \dots np} \right\} \\ &= \frac{m-2}{m} \left\{ \frac{\partial v_{n \dots nij}}{\partial x^n} - 2 \Gamma_{nj}^p v_{n \dots npi} - 2 \Gamma_{in}^p v_{n \dots npj} \right\} + \frac{1}{m} \left\{ \frac{\partial v_{n \dots nj}}{\partial x^i} + \frac{\partial v_{n \dots ni}}{\partial x^j} - 2 \Gamma_{ij}^p v_{n \dots np} \right\}. \end{aligned}$$

From above, we see that the result is true for $k = 0, 1$ and 2 . Now, we are going to prove that the result is also true for $k \leq m$. Consider

$$\begin{aligned} (\mathrm{d}v)_{n \dots ni_k \dots i_1} &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\frac{\partial v_{n \dots ni_k \dots i_2}}{\partial x^{i_1}} - \sum_{l=2}^k \Gamma_{i_l i_1}^p v_{n \dots ni_k \dots \hat{i}_l \dots i_{2p}} - (m-k) \Gamma_{ni_1}^p v_{n \dots ni_k \dots i_{2p}} \right) \\ &= J + J_k^1 + (m-k) J_k^2. \end{aligned}$$

where

$$\begin{aligned}
J &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\frac{\partial v_{n \dots n i_k \dots i_2}}{\partial x^{i_1}} \right), \\
J_k^1 &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\sum_{l=2}^k \Gamma_{i_l i_1}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_{2p}} \right), \\
\text{and } J_k^2 &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\Gamma_{n i_1}^p v_{n \dots n i_k \dots i_{2p}} \right).
\end{aligned}$$

$$\begin{aligned}
J &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\frac{\partial v_{n \dots n i_k \dots i_2}}{\partial x^{i_1}} \right) \\
&= \frac{\sigma(n, \dots, n, i_k, \dots, i_2)}{m} \left(\frac{\partial v_{n \dots n i_k \dots i_2}}{\partial x^{i_1}} + (m-1) \frac{\partial v_{n \dots n i_k \dots i_1}}{\partial x^n} \right) \\
&= \frac{1}{m} \frac{\partial v_{n \dots n i_k \dots i_2}}{\partial x^{i_1}} + \frac{\sigma(n, \dots, n, i_k, \dots, i_3)}{m} \left(\frac{\partial v_{n \dots n i_k \dots i_3 i_1}}{\partial x^{i_2}} + (m-2) \frac{\partial v_{n \dots n i_k \dots i_1}}{\partial x^n} \right) \\
&= \frac{1}{m} \sum_{l=1}^k \frac{\partial v_{n \dots n i_k \dots i_{l-1} i_{l+1} \dots i_1}}{\partial x^{i_l}} + \frac{m-k}{m} \frac{\partial v_{n \dots n i_k \dots i_1}}{\partial x^n}, \quad \text{repeating similar arguments.}
\end{aligned}$$

$$\begin{aligned}
J_k^2 &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\Gamma_{n i_1}^p v_{n \dots n i_k \dots i_{2p}} \right) \\
&= \frac{\sigma(n, \dots, n, i_k, \dots, i_2)}{m} \left(2\Gamma_{n i_1}^p v_{n \dots n i_k \dots i_{2p}} + (m-2)\Gamma_{n i_2}^p v_{n \dots n i_k \dots i_3 i_1 p} \right) \\
&= \frac{2\sigma(n, \dots, n, i_k, \dots, i_3)}{m(m-1)} \left(\Gamma_{i_2 i_1}^p v_{n \dots n i_k \dots i_{3p}} + (m-2)\Gamma_{n i_1}^p v_{n \dots n i_k \dots i_{2p}} \right) \\
&\quad + \frac{(m-2)\sigma(n, \dots, n, i_k, \dots, i_3)}{m(m-1)} \left(2\Gamma_{n i_2}^p v_{n \dots n i_k \dots i_3 i_1 p} + (m-3)\Gamma_{n i_3}^p v_{n \dots n i_k \dots i_4 i_2 i_1 p} \right) \\
&= \frac{2}{m(m-1)} \Gamma_{i_2 i_1}^p v_{n \dots n i_k \dots i_{3p}} + \frac{2(m-2)\sigma(n, \dots, n, i_k, \dots, i_3)}{m(m-1)} \sum_{q=1}^2 \Gamma_{n i_q}^p v_{n \dots n i_k \dots i_3 \hat{i}_q i_{1p}} \\
&\quad + \frac{(m-3)(m-2)\sigma(n, \dots, n, i_k, \dots, i_3)}{m(m-1)} \Gamma_{n i_3}^p v_{n \dots n i_k \dots i_2 i_{1p}} \\
&= \frac{2}{m(m-1)} \sum_{q,r=1, q \neq r}^3 \Gamma_{i_q i_r}^p v_{n \dots n i_k \dots i_4 \hat{i}_q \hat{i}_r i_{1p}} + \frac{2(m-3)\sigma(n, \dots, n, i_k, \dots, i_4)}{m(m-1)} \sum_{q=1}^3 \Gamma_{n i_q}^p v_{n \dots n i_k \dots i_4 \hat{i}_q i_{1p}} \\
&\quad + \frac{(m-4)(m-3)\sigma(n, \dots, n, i_k, \dots, i_4)}{m(m-1)} \Gamma_{n i_4}^p v_{n \dots n i_k \dots i_5 i_3 i_2 i_{1p}},
\end{aligned}$$

repeating similar calculation $(k-3)$ times, we get

$$= \frac{2(k-1)}{m(m-1)} \sum_{q,r=1, q \neq r}^k \Gamma_{i_q i_r}^p v_{n \dots n i_k \dots \hat{i}_q \dots \hat{i}_r \dots i_{1p}} + \frac{2(m-k)}{m(m-1)} \sum_{q=1}^k \Gamma_{n i_q}^p v_{n \dots n i_k \dots \hat{i}_q \dots i_{1p}}.$$

$$\begin{aligned}
J_k^1 &= \sigma(n, \dots, n, i_k, \dots, i_1) \left(\sum_{l=2}^k \Gamma_{i_l i_1}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_2 p} \right) \\
&= \frac{\sigma(n, \dots, n, i_k, \dots, i_2)}{m} \left(2 \sum_{l=2}^k \Gamma_{i_l i_1}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_2 p} + (m-2) \sum_{l=3}^k \Gamma_{i_l i_2}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_3 i_1 p} \right. \\
&\quad \left. + (m-2) \Gamma_{i_3 i_2}^p v_{n \dots n i_k \dots i_4 i_1 p} \right) \\
&= \frac{\sigma(n, \dots, n, i_k, \dots, i_3)}{m(m-1)} \left\{ 2(k-1) \Gamma_{i_2 i_1}^p v_{n \dots n i_k \dots i_3 p} + (m-2) \left(2 \sum_{l=3}^k \Gamma_{i_l i_1}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_2 p} \right. \right. \\
&\quad \left. \left. + 2 \Gamma_{i_3 i_1}^p v_{n \dots n i_k \dots i_4 i_2 p} + 2 \sum_{l=3}^k \Gamma_{i_l i_2}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_3 i_1 p} + (m-3) \sum_{l=4}^k \Gamma_{i_l i_3}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_4 i_2 i_1 p} \right. \right. \\
&\quad \left. \left. + (m-3) \Gamma_{i_3 i_4}^p v_{n \dots n i_k \dots i_5 i_2 i_1 p} + 2 \Gamma_{i_3 i_2}^p v_{n \dots n i_k \dots i_4 i_1 p} + (m-3) \Gamma_{i_3 i_4}^p v_{n \dots n i_k \dots i_5 i_2 i_1 p} \right) \right\} \\
&= \frac{(m-2)\sigma(n, \dots, n, i_k, \dots, i_3)}{m(m-1)} \left\{ 2 \sum_{q=1}^2 \left(\sum_{l=3}^k \Gamma_{i_l i_q}^p v_{n \dots n i_k \dots \hat{i}_l \dots \hat{i}_q i_1 p} + \Gamma_{i_3 i_q}^p v_{n \dots n i_k \dots i_4 \hat{i}_q i_1 p} \right) \right. \\
&\quad \left. + (m-3) \left(\sum_{l=4}^k \Gamma_{i_l i_3}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_4 i_2 i_1 p} + 2 \Gamma_{i_3 i_4}^p v_{n \dots n i_k \dots i_5 i_2 i_1 p} \right) \right\} + \frac{2(k-1)}{m(m-1)} \Gamma_{i_2 i_1}^p v_{n \dots n i_k \dots i_3 p} \\
&= \frac{(m-3)\sigma(n, \dots, n, i_k, \dots, i_4)}{m(m-1)} \left\{ 2 \sum_{q=1}^3 \left(\sum_{l=4}^k \Gamma_{i_l i_q}^p v_{n \dots n i_k \dots \hat{i}_l \dots \hat{i}_q i_1 p} \right) + 4 \sum_{q=1}^3 \left(\Gamma_{i_4 i_q}^p v_{n \dots n i_k \dots \hat{i}_q i_1 p} \right) \right. \\
&\quad \left. + (m-4) \left(\sum_{l=5}^k \Gamma_{i_l i_4}^p v_{n \dots n i_k \dots \hat{i}_l \dots i_1 p} + 3 \Gamma_{i_5 i_4}^p v_{n \dots n i_k \dots i_1 p} \right) \right\} + \frac{2(k-1)}{m(m-1)} \sum_{q,r=1, q \neq r}^3 \Gamma_{i_q i_r}^p v_{n \dots n i_k \dots \hat{i}_q \hat{i}_r i_1 p}
\end{aligned}$$

Repeating this expansion for $(k-2)$ times more to get

$$\begin{aligned}
J_k^1 &= \frac{(m-k+1)\sigma(n, \dots, n, i_k)}{m(m-1)} \left\{ 2 \sum_{q=1}^{k-1} \Gamma_{i_k i_q}^p v_{n \dots n i_{k-1} \dots \hat{i}_q \dots i_1 p} + 2(k-2) \sum_{q=1}^{k-1} \Gamma_{i_k i_q}^p v_{n \dots n i_{k-1} \dots \hat{i}_q \dots i_1 p} \right. \\
&\quad \left. + (m-k)(k-1) \Gamma_{n i_k}^p v_{n \dots n i_{k-1} \dots i_1 p} \right\} + \frac{2(k-1)}{m(m-1)} \sum_{q,r=1, q \neq r}^{k-1} \Gamma_{i_q i_r}^p v_{n \dots n i_k \dots \hat{i}_q \hat{i}_r i_1 p} \\
&= \frac{(m-k+1)\sigma(n, \dots, n, i_k)}{m(m-1)} \left\{ 2(k-1) \sum_{q=1}^{k-1} \Gamma_{i_k i_q}^p v_{n \dots n i_{k-1} \dots \hat{i}_q \dots i_1 p} \right. \\
&\quad \left. + (m-k)(k-1) \Gamma_{n i_k}^p v_{n \dots n i_{k-1} \dots i_1 p} \right\} + \frac{2(k-1)}{m(m-1)} \sum_{q,r=1, q \neq r}^{k-1} \Gamma_{i_q i_r}^p v_{n \dots n i_k \dots \hat{i}_q \hat{i}_r i_1 p} \\
&= \frac{2(k-1)}{m(m-1)} \sum_{q,r=1, q \neq r}^k \Gamma_{i_q i_r}^p v_{n \dots n i_k \dots \hat{i}_q \dots \hat{i}_r \dots i_1 p} + \frac{2(k-1)(m-k)}{m(m-1)} \sum_{q=1}^k \Gamma_{n i_q}^p v_{n \dots n i_k \dots \hat{i}_q \dots i_1 p}
\end{aligned}$$

After putting the values of J, J_k^1 and J_k^2 in dv , we get

$$\begin{aligned} (dv)_{n\dots ni_k\dots i_1} &= \frac{(m-k)}{m} \frac{\partial v_{n\dots ni_k\dots i_1}}{\partial x^n} - \frac{2(m-k)}{m} \sum_{l=1}^k \Gamma_{i_l n}^p v_{n\dots ni_k\dots \hat{i}_l\dots i_{1p}} \\ &\quad + \frac{1}{m} \sum_{l=1}^k \frac{\partial v_{n\dots ni_k\dots \hat{i}_l\dots i_1}}{\partial x^{i_l}} - \frac{2}{m} \sum_{l,q=1, l \neq q}^k \Gamma_{i_l i_q}^p v_{n\dots ni_k\dots \hat{i}_l\dots \hat{i}_q\dots i_{1p}}. \end{aligned}$$

□

Proof of estimate (11). Let ${}^t L = \frac{\Phi_\xi \cdot \partial_\xi}{i\lambda |\Phi_\xi|^2}$. Then as already noted

$${}^t L^N(e^{i\lambda\Phi(x,\xi)}) = e^{i\lambda\Phi(x,\xi)}.$$

Consider,

$$\begin{aligned} &\left| \iint_{|x-y|>\delta/C_0} ({}^t L^N(e^{i\lambda\Phi(y,x,\xi,\eta)})) \tilde{a}_N(x,\xi) f_{i_1\dots i_m}(z) \tilde{b}^{i_1}(x,\xi) \dots \tilde{b}^{i_m}(x,\xi) dx d\xi \right| \\ &\leq \left| \iint_{|x-y|>\delta/C_0} e^{i\lambda\Phi(y,x,\xi,\eta)} L^N(\tilde{a}_N(x,\xi) f_{i_1\dots i_m}(z) \tilde{b}^{i_1}(x,\xi) \dots \tilde{b}^{i_m}(x,\xi)) dx d\xi \right| \\ &\quad + N \iint_{|x-y|\geq\delta/C_0} e^{-\lambda\delta^2/2} f_{i_1\dots i_m}(x) B^{i_1\dots i_m}(x, \xi_{bdry}) dx. \end{aligned}$$

Using the fact that, f is compactly supported and using (7), we get (11). □

Proof of the estimate (13). Consider

$$\left| \iint_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi,\eta)}) \tilde{a}_N(x,\xi) f_{i_1\dots i_m}(z) \tilde{b}^{i_1}(x,\xi) \dots \tilde{b}^{i_m}(x,\xi) dx d\xi \right|$$

Rewrite the above as :

$$\left| \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) (e^{-i\lambda\Phi(y,x,\xi_c,\eta)}) \int_{|\xi-\eta|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi,\eta)}) \tilde{a}_N(x,\xi) f_{i_1\dots i_m}(z) \tilde{b}^{i_1}(x,\xi) \dots \tilde{b}^{i_m}(x,\xi) dx d\xi \right|$$

Using (2.10) of [18] to the above, we get

$$\begin{aligned} &\left| \int_{|x-y|<\delta/C_0} (e^{i\lambda\Phi(y,x,\xi_c,\eta)}) f_{i_1\dots i_m}(x) \sum_{0 \leq k \leq \lambda/C} C_n \frac{1}{k!} \lambda^{-n/2-k} \frac{(\Delta^k)}{2} (\tilde{a}_N(x, \xi_c) \tilde{b}^{i_1}(x, \xi_c) \dots \tilde{b}^{i_m}(x, \xi_c)) \right. \\ &\quad \left. + R(x, y, \eta, \lambda) dx \right| \end{aligned}$$

Lemma 8.

$$\sum_{0 \leq k \leq \lambda/C} C_n \frac{1}{k!} \lambda^{-n/2-k} \frac{(\Delta^k)}{2} (\tilde{a}_N(x, \xi_c) \tilde{b}^{i_1}(x, \xi_c) \dots \tilde{b}^{i_m}(x, \xi_c))$$

is a formal analytic symbol.

Proof. Let,

$$a_k = \frac{1}{k!} \frac{(\Delta^k)}{2} (\tilde{a}_N(x, \xi_c) \tilde{b}^{i_1}(x, \xi_c) \cdots \tilde{b}^{i_m}(x, \xi_c))$$

Then from Cauchy integral formula [18, Section 2.4],

$$\begin{aligned} |a_k| &\leq C_n (k+1)^{n/2} (k-1)! 2^k \sup_{B(\xi_c)} (\tilde{a}_N(x, \xi_c) \tilde{b}^{i_1}(x, \xi_c) \cdots \tilde{b}^{i_m}(x, \xi_c)) \\ &\leq C 1_n (k+1)^{n/2} (k-1)! 2^k \\ &\leq C 2_n (k+1)^{n/2} e^{2-k} (k-1)^{k-1/2} 2^k \quad (\text{Using Stirling's approximation}) \\ &\leq C 2_n \left(\frac{2}{e}\right)^{k+1} (k+1)^{n/2+k} \\ &\leq \tilde{C}_n^{k+n/2} (k+n/2)^{n/2+k} \end{aligned}$$

Hence,

$$\sum_{0 \leq k \leq \lambda/C} C_n \frac{1}{k!} \lambda^{-n/2-k} \frac{(\Delta^k)}{2} (\tilde{a}_N(x, \xi_c) \tilde{b}^{i_1}(x, \xi_c) \cdots \tilde{b}^{i_m}(x, \xi_c)) = \sum_{0 \leq k \leq \lambda/C} \lambda^{-n/2-k} a_{k+n/2}$$

is a formal analytic symbol $B^{i_1 \dots i_m}(x, y, \eta; \lambda)$ by [18, ex 1.1] □

Hence,

$$\begin{aligned} &\int_{|x-y| < \delta/C_0} (e^{i\lambda\Phi(y, x, \xi, \eta)}) \tilde{a}_N(x, \xi) f_{i_1 \dots i_m}(z) \tilde{b}^{i_1}(x, \xi) \cdots \tilde{b}^{i_m}(x, \xi) dx d\xi \\ &= \int_{|x-y| < \delta/C_0} (e^{i\lambda\Phi(y, x, \xi_c, \eta)}) f_{i_1 \dots i_m}(x) B^{i_1 \dots i_m}(x, y, \eta; \lambda) dx \\ &\quad + \int_{|x-y| < \delta/C_0} (e^{i\lambda\Phi(y, x, \xi_c, \eta)}) f_{i_1 \dots i_m}(x) R(x, y, \eta; \lambda) dx d\xi \end{aligned}$$

But,

$$\left| \int_{|x-y| < \delta/C_0} (e^{i\lambda\Phi(y, x, \xi_c, \eta)}) f_{i_1 \dots i_m}(x) R(x, y, \eta; \lambda) dx \right| = \mathcal{O}(e^{-\lambda/c}).$$

Since,

$$|R(x, y, \eta; \lambda)| \leq \Omega/C e^{-\lambda/c}$$

(See 2.10, [18]). So, this along with (10) and (11), gives us:

$$\left| \int_{|x-y| < \delta/C_0} (e^{i\lambda\Phi(y, x, \xi_c, \eta)}) f_{i_1 \dots i_m}(x) B^{i_1 \dots i_m}(x, y, \eta; \lambda) dx \right| = \mathcal{O}(e^{-\lambda/c}).$$

□

References Cited

- [1] Jan Boman. Helgason’s support theorem for Radon transforms—a new proof and a generalization. In *Mathematical methods in tomography (Oberwolfach, 1990)*, volume 1497 of *Lecture Notes in Math.*, pages 1–5. Springer, Berlin, 1991.
- [2] Jan Boman. Holmgren’s uniqueness theorem and support theorems for real analytic Radon transforms. In *Geometric analysis (Philadelphia, PA, 1991)*, volume 140 of *Contemp. Math.*, pages 23–30. Amer. Math. Soc., Providence, RI, 1992.
- [3] Jan Boman and Eric Todd Quinto. Support theorems for real-analytic Radon transforms. *Duke Math. J.*, 55(4):943–948, 1987.
- [4] Jan Boman and Eric Todd Quinto. Support theorems for Radon transforms on real analytic line complexes in three-space. *Trans. Amer. Math. Soc.*, 335(2):877–890, 1993.
- [5] Gregory Eskin. Inverse scattering problem in anisotropic media. *Comm. Math. Phys.*, 199(2):471–491, 1998.
- [6] Fulton Gonzalez and Eric Todd Quinto. Support theorems for Radon transforms on higher rank symmetric spaces. *Proc. Amer. Math. Soc.*, 122(4):1045–1052, 1994.
- [7] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.
- [8] Lars Hörmander. *The analysis of linear partial differential operators. I*. Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].
- [9] Venkateswaran P. Krishnan. A support theorem for the geodesic ray transform on functions. *J. Fourier Anal. Appl.*, 15(4):515–520, 2009.
- [10] Venkateswaran P. Krishnan and Rohit Kumar Mishra. Microlocal analysis for restricted ray transform of symmetric tensor field in n-dimension. In Preparation.
- [11] Venkateswaran P. Krishnan and Plamen Stefanov. A support theorem for the geodesic ray transform of symmetric tensor fields. *Inverse Problems and Imaging*, 3(3):453–464, 2009.
- [12] C. B. Morrey, Jr. and L. Nirenberg. On the analyticity of the solutions of linear elliptic systems of partial differential equations. *Comm. Pure Appl. Math.*, 10:271–290, 1957.
- [13] Eric Todd Quinto. Support Theorems for the Spherical Radon Transform on Manifolds. *International Mathematics Research Notices*, 2006:1–17, 2006. Article ID = 67205.

- [14] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara. Microfunctions and pseudo-differential equations. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, pages 265–529. Lecture Notes in Math., Vol. 287. Springer, Berlin, 1973.
- [15] V. A. Sharafutdinov. ray transform on riemannian manifolds, lecture notes. UWSeattle available at: <http://www.ima.umn.edu/talks/workshops/7-16-27.2001/sharafutdinov/>, 1999.
- [16] V. A. Sharafutdinov. A problem of integral geometry for generalized tensor fields on \mathbf{R}^n . *Dokl. Akad. Nauk SSSR*, 286(2):305–307, 1986.
- [17] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.
- [18] Johannes Sjöstrand. Singularités analytiques microlocales. In *Astérisque*, 95, volume 95 of *Astérisque*, pages 1–166. Soc. Math. France, Paris, 1982.
- [19] Plamen Stefanov and Gunther Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.
- [20] Plamen Stefanov and Gunther Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003 (electronic), 2005.
- [21] Plamen Stefanov and Gunther Uhlmann. Integral geometry on tensor fields on a class of non-simple Riemannian manifolds. *Amer. J. Math.*, 130(1):239–268, 2008.
- [22] Gunther Uhlmann and András Vasy. The inverse problem for the local geodesic ray transform. *Inventiones Mathematicae*, 205(1):83–120, 2016.
- [23] Yiyang Zhou and Eric Todd Quinto. Two-radius support theorems for spherical Radon transforms on manifolds. In *Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998)*, volume 251 of *Contemp. Math.*, pages 501–508. Amer. Math. Soc., Providence, RI, 2000.